3

Time Domain Laser Beam Propagation

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This chapter reviews the propagation of time domain pulses through optical media, such as glasses, optical fibers, or nonlinear crystals, and special optical systems, such as stretchers or compressors, which are designed to modify time domain pulses in a very specific way. The chapter starts with the wave equation as it is commonly used in optics and as introduced in Chapter 1. The material equations are derived from a classical model of a driven oscillator in an anharmonic potential. Despite the simplicity of the model, it is sufficient to qualitatively explain a great number of effects in linear as well as in nonlinear optics. The entire chapter will be based on a one-dimensional propagation equation, that is, the results are applicable whenever the transverse beam size remains more or less constant as the pulse propagates through the dielectric material. We then proceed with more complex linear systems, which are typically composed of various optical elements and act on the temporal or spectral part of the electric field. To further illustrate the different effects, we show simulation results for 100-fs-long and 10-fs-long Gaussian-shaped time domain pulses, respectively; their center wavelength is
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800 nm. All the simulations in this chapter have been performed with the Lab2 simulation software [1].

### 3.1 Wave Equation

From Maxwell’s equations in coordinate space and time domain as defined in Chapter 1 and assuming dielectric media, the wave equation

\[
\nabla^2 E(x, t) - \frac{1}{c^2} \frac{\partial^2 E(x, t)}{\partial t^2} - \mu_0 \frac{\partial^2 P(x, t)}{\partial t^2} = 0 \tag{3.1}
\]

can be derived. Assuming that the electric field can be written as a product \( E(x, t) = f(x, y) E(z, t) \) and that the transverse beam support is such that the transverse derivatives of \( f(x, y) \) may be safely neglected, the wave equation in time domain becomes

\[
\frac{\partial^2 E(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E(z, t)}{\partial t^2} - \mu_0 \frac{\partial^2 P(z, t)}{\partial t^2} = 0 \tag{3.2}
\]

and in frequency domain

\[
\frac{\partial^2 E(z, \omega)}{\partial z^2} + \frac{\omega^2}{c^2} E(z, \omega) + \omega^2 \mu_0 P(z, \omega) = 0. \tag{3.3}
\]

To proceed, we employ a simple classical picture to find a constituent relationship between the polarization and the electric field in dielectric media, that is, \( P(E) \). We assume that the dielectric material consists of atomic-scale subsystems that can couple to an external electric field through their dipole moment. Once accelerated, they start to radiate and act as sources of radiation. Further, we assume that the subsystems are independent, that is do not couple to each other, and are sufficiently well described by driven oscillators in a generally anharmonic potential. For example, such subsystems may be electrons bound to ions or optical phonons. When the driving electric field is strong enough, the oscillators will leave the mostly harmonic part and start to explore the increasingly anharmonic parts of the potential surface. To employ a perturbation approach to solve the oscillator’s equation of motion, it is convenient to expand the potential energy surface in a Taylor series [2]. The material’s polarization is related to the oscillator position vector \( s \) through \( P_i(z, t) = N_i q_i s_i(t) \) with the number of oscillators \( N_i \) and their charge \( q_i \) and we find

\[
\frac{\partial^2 P_i(z, t)}{\partial t^2} + \gamma_i \frac{\partial P_i(z, t)}{\partial t} + \omega_0^2 P_i(z, t) + \sum_{j,k} a_{ijk} P_j(z, t) P_k(z, t) + \sum_{jkl} b_{ijkl} P_j(z, t) P_k(z, t) P_l(z, t) + \cdots = \varepsilon_0 f_i E_i(z, t), \tag{3.4}
\]
Equation 3.4 is solved by inserting a power series expansion for the polarization, that is

\[ P_i(z, t) = P_i^{(0)}(z, t) + \lambda P_i^{(1)}(z, t) + \lambda^2 P_i^{(2)}(z, t) + \lambda^3 P_i^{(3)}(z, t) + \cdots, \]  

(3.5)

where \( \lambda \) is related to the strength of the perturbation. The original perturbation \( \epsilon_0 f_i E_i(z, t) \) is replaced by \( \lambda \epsilon_0 f_i E_i(z, t) \). The power series Ansatz is a solution to Equation 3.4 if all terms proportional to \( \lambda^n \) satisfy the equation separately. The resulting system of equations is now solved in order until reaching some \( n \) after which the remaining terms may be safely neglected. In the following, we assume that the oscillator starts at rest, which means that \( P_i^{(0)}(z, t) \) must be zero for all times. For molecules, this is not always true as they frequently have a static dipole moment. Collecting orders up to \( n = 3 \) and transforming to the frequency domain yields

\[ P_i^{(1)}(z, \omega) = \frac{\epsilon_0}{2\pi} \sum_{j,k} \int d\omega_2 \chi_{ijk}^{(2)}(\omega_1, \omega_2, \omega_2 - \omega_2) E_j(z, \omega_2) E_k(z, \omega - \omega_2) \]  

(3.6a)

\[ P_i^{(2)}(z, \omega) = \frac{\epsilon_0}{4\pi^2} \sum_{j,k,l} \int\int d\omega_2 d\omega_3 \chi_{ijkl}^{(3)}(\omega_1, \omega_2, \omega_3, \omega - \omega_2 - \omega_3) 
\times E_j(z, \omega_2) E_k(z, \omega_3) E_l(z, \omega - \omega_2 - \omega_3), \]  

(3.6b)

(3.6c)

where we have used the definitions

\[ \chi_{ii}^{(1)}(\omega; \omega) \doteq \frac{f_i}{\omega_i^2 - \omega^2 + i\omega\gamma_i} \]  

(3.7a)

\[ \chi_{ijk}^{(2)}(\omega_1, \omega_2, \omega - \omega_2) \doteq -\frac{a_{ijk}\epsilon_0}{f_i} \chi_{ii}^{(1)}(\omega; \omega) \chi_{jj}^{(1)}(\omega_2; \omega_2) \times \chi_{kk}^{(1)}(\omega - \omega_2; \omega - \omega_2) \]  

(3.7b)

\[ \chi_{ijkl}^{(3)}(\omega_1, \omega_2, \omega_3, \omega - \omega_2 - \omega_3) \doteq \frac{\epsilon_0}{f_i} \left[ \sum_m 2a_{ijm}a_{mkl} \chi_{mm}^{(1)}(\omega - \omega_2; \omega - \omega_2) - b_{ijkl} \right] \times \chi_{ii}^{(1)}(\omega; \omega) \chi_{jj}^{(1)}(\omega_2; \omega_2) \chi_{kk}^{(1)}(\omega_3; \omega_3) \times \chi_{ll}^{(1)}(\omega - \omega_2 - \omega_3; \omega - \omega_2 - \omega_3). \]  

(3.7c)

Because of energy conservation, the first frequency argument in \( \chi^{(n)} \) must be equal to the sum of all the following frequencies. Note that in this classical picture, all higher-order susceptibilities can be related to a combination of first-order susceptibilities. Most optical materials possess more than just a single type of oscillator and the first-order susceptibility (Equation 3.7a) turns into a sum over several resonances. While phonon resonances typically appear in the THz to IR part of the spectrum, electronic resonances are found in
the blue to UV part. Glasses and crystals are probably the most important dielectric materials in optics and all their resonances are typically far outside the visible spectral region and, consequently, the imaginary part of the susceptibility may be safely neglected. For not-too-short pulses and for frequencies far away from the resonance frequencies $\omega_0$, the dispersive properties of the second- and third-order susceptibilities may be safely neglected. With this, $\chi^{(2)}$ and $\chi^{(3)}$ become constant and their Fourier transforms are $R^{(2)}(t_1, t_2) = \chi^{(2)} \delta(t_1) \delta(t_2)$ and $R^{(3)}(t_1, t_2, t_3) = \chi^{(3)} \delta(t_1) \delta(t_2) \delta(t_3)$, respectively. Then the solutions (3.6a) simplify to

$$P_i^{(1)}(z, t) = \varepsilon_0 \int dt_1 R^{(1)}_{ii}(t_1) E_i(z, t - t_1)$$

(3.8a)

$$P_i^{(2)}(z, t) = \varepsilon_0 \sum_{j,k} \chi^{(2)}_{ijk} E_j(z, t) E_k(z, t)$$

(3.8b)

$$P_i^{(3)}(z, t) = \varepsilon_0 \sum_{j,k,l} \chi^{(3)}_{ijkl} E_j(z, t) E_k(z, t) E_l(z, t).$$

(3.8c)

Inserting Equations 3.8 for the polarization yields the wave equation in the time domain

$$\partial_z^2 E(z, t) = \frac{1}{c^2} \partial_t^2$$

$$\times \left[ \int dt_1 \mathbf{e}_r(t_1) \cdot \mathbf{E}(z, t - t_1) + \chi^{(2)} : \mathbf{E}(z, t) \cdot \mathbf{E}(z, t) + \chi^{(3)} : \mathbf{E}(z, t) \cdot \mathbf{E}(z, t) \cdot \mathbf{E}(z, t) \right],$$

(3.9)

with the relative permittivity $\varepsilon_{ii}(t) = \delta(t) + R^{(1)}_{ii}(t)$. Similarly, in the frequency domain, we obtain

$$\partial_z^2 E(z, \omega) = -\frac{\omega^2}{c^2} \left[ \mathbf{e}_r(\omega) \cdot \mathbf{E}(z, \omega) + \frac{1}{2\pi} \int d\omega_1 \chi^{(2)} : \mathbf{E}(z, \omega_1) \cdot \mathbf{E}(z, \omega - \omega_1) \right.$$

$$\left. + \frac{1}{4\pi^2} \int d\omega_1 d\omega_2 \chi^{(3)} : \mathbf{E}(z, \omega_2) \cdot \mathbf{E}(z, \omega_1 - \omega_2) \cdot \mathbf{E}(z, \omega - \omega_1) \right],$$

(3.10)

with $\varepsilon_{ii}(\omega) = 1 + \chi^{(1)}_{ii}(\omega)$. There is no clear preference whether to solve the nonlinear wave equation in the time or the frequency domain, as in both cases, convolutions appear. In practice, we mostly encounter second- or third-order effects separately, which is why we treat them independently.
3.2 Solution to the Linear Wave Equation

Neglecting all nonlinear contributions to the polarization brings us to the realms of linear optics. The wave equation simplifies to

$$\frac{\partial^2}{\partial z^2} E(z, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int dt_1 \, \epsilon_r(t_1) \cdot E(z, t - t_1).$$

(3.11)

Generally, the relative permittivity $\epsilon_r$ is a tensor of rank two and simplifies to a scalar for isotropic materials, such as glasses or liquids [3]. Moreover, we can always rotate the coordinate system such that the tensor becomes diagonal. The index of refraction of an optical material is defined through

$$n_{ii}(\omega) = \sqrt{\epsilon_{ii}(\omega)}. \quad \text{(3.12)}$$

That is, a material’s linear response to an external electric field may either be described through its susceptibility, its dielectric function, or its index of refraction. For frequencies where $\partial_\omega \Re\{\epsilon_r(\omega)\} > 0$, the material is said to have normal dispersion and for $\partial_\omega \Re\{\epsilon_r(\omega)\} < 0$, the material shows anomalous dispersion. The linear wave equation is readily solved and the most general solution presents itself as a superposition of plane waves propagating in the positive $z$-direction

$$E_i(z, t) = \frac{1}{2\pi} \int d\omega \ E_i(z, \omega) \ e^{i\omega t - ik_i(\omega)z}.$$  

(3.13)

Each plane wave component is characterized by an amplitude, a frequency, and a wave number $k_i$ that is related to the plane wave’s frequency $\omega$ through the dispersion relation

$$k_i(\omega) = \frac{\omega}{c} \ n_{ii}(\omega). \quad \text{(3.14)}$$

In the following, we use the phasor field description as introduced in Chapter 1, that is

$$E(z, t) = E^+(z, t) + E^-(z, t),$$

(3.15)

with $E^-(z, t) = [E^+(z, t)]^*$. Frequently, the spectral support of a pulse is limited to a narrow region around the carrier frequency $\omega_0$. Then, we often employ the slowly varying envelope form

$$E^+(z, t) = \frac{1}{2} \ E^+(z, t) \ e^{i\omega_0 t - ik_0 z}. \quad \text{(3.16)}$$

With Equations 3.13 and 3.16, it is obvious that the slowly varying field at some distance $z$ is related to the field at $z = 0$ through

$$E_i^+(z, t) = \frac{1}{2\pi} \int d\Omega \ E_i^+(\Omega) \ e^{i\Omega t - ik_i(\omega)z}, \quad \text{(3.17)}$$
with $\mathcal{E}_i^+(\Omega) \doteq \mathcal{E}_i^+(z = 0, \Omega)$ being the Fourier transform of the pulse $\mathcal{E}_i^+(z = 0, t)$ and the relative frequency $\Omega \doteq \omega - \omega_0$. If the index of refraction is a relatively smooth function within the region of interest, the wave number may be expanded in a Taylor series and it is sufficient to consider only low-order Taylor coefficients, that is

$$k(\omega) = \sum_j \frac{1}{j!} \left. \frac{d^j k(\omega)}{d\omega^j} \right|_{\omega_0} (\omega - \omega_0)^j. \tag{3.18}$$

As a rule of thumb, if

$$\frac{1}{j!} \left. \frac{d^j k(\omega)}{d\omega^j} \right|_{\omega_0} \Delta \omega^j z \ll 1 \tag{3.19}$$

holds, with $\Delta \omega$ being the spectral width of the pulse, then the corresponding orders may be safely neglected. The phase velocity is related to the zero-order term through

$$\frac{1}{v_p} = \frac{k_0}{\omega_0} = \frac{n(\omega_0)}{c}, \tag{3.20}$$

where we have used $k_0 \doteq k(\omega_0)$. The group velocity is defined through

$$\frac{1}{v_g} = \frac{dk}{d\omega} \bigg|_{\omega_0} = \frac{1}{c} \left[ n(\omega_0) + \omega_0 \frac{dn}{d\omega} \bigg|_{\omega_0} \right] = \frac{1}{c} \left[ n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda} \bigg|_{\lambda_0} \right] = \frac{n_g(\omega_0)}{c}, \tag{3.21}$$

with $n_g$ being the group index. The group velocity dispersion (GVD) is defined through

$$D_\omega \doteq \frac{d^2 k}{d\omega^2} \bigg|_{\omega_0}. \tag{3.22}$$

In optical communication, the dispersion parameter $D_\lambda$ is more frequently used

$$D_\lambda \doteq -\frac{2\pi c}{\lambda_0^2} D_\omega = -\frac{2\pi c}{\lambda_0^2} \left. \frac{d^2 k}{d\omega^2} \right|_{\omega_0}. \tag{3.23}$$

Terminating the series expansion (3.18) at $j = 2$ yields

$$\mathcal{E}_i^+(z, t) = \frac{1}{2\pi} \int d\Omega \mathcal{E}_0^+(\Omega) \exp \left[ i \left( t - \frac{z}{v_g} \right) \Omega - i \frac{D_\omega z^2}{2} \right]. \tag{3.24}$$

Often, we are interested in changes of the shape of the pulse and not the propagation itself. Then it is useful to transform to a coordinate system, which moves with the group velocity of the pulse. In this coordinate system, the new coordinates are $\zeta = z$
and \( \tau = t - z/v_g \) and we obtain

\[
\mathcal{E}^+(\zeta, \tau) = \frac{1}{2\pi} \int d\Omega \mathcal{E}^0_0(\Omega) \exp \left[ i\Omega \tau - i\frac{D\omega\zeta^2}{2} \right].
\] (3.25)

Figure 3.1 shows two pulses with a duration of 100 and 10-fs, respectively, before and after propagation through a 10-mm-thick piece of BK7 glass (e.g., a vacuum window). While the longer pulse is almost unaffected, the 10-fs pulse shows a considerable broadening, which is mostly due to GVD, but also an asymmetry, which is due to higher-order Taylor coefficients not considered in Equation 3.25. Generally, dispersion effects become more of an issue with shorter pulses.

### 3.3 Solution to the Nonlinear Wave Equation

Nonlinear effects in ultrafast optics are often explored to change the spectral content of the original laser pulse, for example, in second harmonic or super continuum generation. Nonlinear effects are also prerequisites for soliton formation in optical fibers and are at the heart of all short-pulse characterization methods. In the following, we restrict the review on nonlinear effects to those relevant for ultrafast optics. We address second- and third-order effect separately, because they mostly appear separately.

#### 3.3.1 Second-Order Nonlinearities: Three Wave Mixing

In the first section, we consider second-order nonlinear effects only, which reduces the wave equation in the frequency domain to

\[
\frac{\partial^2}{\partial z^2} E(z, \omega) + \frac{\omega^2}{c^2} \epsilon_\omega(\omega) \cdot E(z, \omega) = -\frac{\omega^2}{2\pi c^2} \int d\omega_1 \chi^{(2)} : E(z, \omega_1) \cdot E(z, \omega - \omega_1). \] (3.26)
The nonlinearity is such that two fields within the convolution may mix to produce a third, which is why second-order nonlinear effects are also labeled three wave mixing. That is, the total field can be assumed to be a superposition of three fields, which are centered around frequencies $\omega_{0\alpha}$ and which may be polarized in different directions $\hat{e}_\alpha$. In order to simplify, we invoke the following Ansatz:

$$E(z, \omega) = \frac{1}{2} \sum_{\alpha=1}^{3} \left[ \mathcal{E}_\alpha^+(z, \omega - \omega_{0\alpha}) + \mathcal{E}_\alpha^-(z, \omega + \omega_{0\alpha}) \right] e^{-i k_\alpha z} \hat{e}_\alpha. \quad (3.27)$$

To arrive at the next expression, we first insert Equation 3.27 in Equation 3.26, treat all fields $\mathcal{E}_\alpha^\pm$ on the left-hand side of the equation separately assuming that they are all well separated on the frequency axis, invoke the slowly varying approximation $\partial_z \mathcal{E}_\alpha^\pm \ll k_\alpha \partial_z \mathcal{E}_\alpha^\pm$, and multiply the resulting equation from the left with $\hat{e}_\alpha$.

$$\partial_z \mathcal{E}_\alpha^+(z, \omega - \omega_{0\alpha}) = -i \frac{\omega}{8\pi c \alpha} \sum_{\beta, \gamma} \hat{e}_\alpha \cdot \chi^{(2)} : \hat{e}_\beta : \hat{e}_\gamma \int d\omega_1 \left\{ \mathcal{E}_\beta^+(z, \omega_1 - \omega_{0\beta}) \mathcal{E}_\gamma^+(z, \omega - \omega_{0\gamma}) + \mathcal{E}_\beta^+(z, \omega_1 - \omega_{0\beta}) \mathcal{E}_\gamma^-(z, \omega - \omega_1) \\
+ \mathcal{E}_\beta^-(z, \omega_1 + \omega_{0\beta}) \mathcal{E}_\gamma^+(z, \omega - \omega_{0\gamma}) + \mathcal{E}_\beta^-(z, \omega_1 + \omega_{0\beta}) \mathcal{E}_\gamma^-(z, \omega - \omega_{0\gamma}) \right\} e^{ik_\alpha(z-\omega_1)z-i k_\beta(\omega_1)z-ik_\gamma(\omega_{0\gamma})z}. \quad (3.28)$$

All slowly varying field envelopes have nonzero contributions only when the frequency argument is close to zero. Therefore, the four terms on the right-hand side of Equation 3.28 do contribute only if certain frequency relations hold. From the left-side, we deduce that $\omega \approx \omega_{0\alpha}$ must hold, and for the right-hand side, we find similar arguments for the four terms, that is

<table>
<thead>
<tr>
<th>Field $\beta$</th>
<th>Field $\gamma$</th>
<th>$\omega_1 \approx \omega_{0\beta}$</th>
<th>$\omega_0 \alpha - \omega_{0\beta} - \omega_{0\gamma} \approx 0$</th>
<th>$\omega_0 \alpha \approx \omega_{0\beta} + \omega_{0\gamma}$</th>
<th>$\omega_0 \alpha \approx -\omega_{0\beta} - \omega_{0\gamma}$</th>
<th>$\omega_0 \alpha \approx -\omega_{0\beta} - \omega_{0\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1 \approx \omega_{0\beta}$</td>
<td>$\omega_0 \alpha - \omega_{0\beta} + \omega_{0\gamma} \approx 0$</td>
<td>DFG</td>
<td>SFG</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1 \approx -\omega_{0\beta}$</td>
<td>$\omega_0 \alpha + \omega_{0\beta} - \omega_{0\gamma} \approx 0$</td>
<td>DFG</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first term contributes if the sum of the two center frequencies of the two fields on the right-hand side is equal to the center frequency of the field on the left-hand side. Such a process is called sum-frequency generation (SFG). Similarly, the two following terms result in a difference of frequencies, which is why these contributions are labeled difference-frequency generation (DFG). The fourth term apparently does not contribute at all. These conditions are somewhat reminiscent of energy conservation. If all three center frequencies are identical and nonzero, none of these conditions can be fulfilled. Therefore, in second-order nonlinear processes, or three wave mixing, we usually have
at least two different fields, typically three, with different center frequencies involved. In
order to proceed and to arrive at somewhat more practical expressions, we need to specify
in detail the process of interest. In the following, we select SFG as a practical example.

In SFG, two low-frequency input fields generate a third, high-frequency field. Energy
conservation requires $\omega_{03} \approx \omega_{01} + \omega_{02}$. We consider a negative uniaxial crystal and the
phase matching is type I (ooe) [4]. That is, $\mathcal{E}_1^\pm$ and $\mathcal{E}_2^\pm$ are polarized in the $x$
direction (ordinary) and $\mathcal{E}_3^\pm$ is polarized along the $y$ direction (extraordinary). Note that for every
field on the left-hand side of Equation 3.28, there are two contributions on the right-hand
side corresponding to different combinations of $\beta$ and $\gamma$. By use Kleinman’s conjecture,
introducing relative frequencies, and assuming a loss-less medium, we arrive at the final
expressions

$$
\partial_z \mathcal{E}_1^+(z, \Omega_1) = -i\kappa_1 \int d\Omega_3 \ \mathcal{E}_2^-(z, \Omega_1 - \Omega_3) \ \mathcal{E}_3^+(z, \Omega_3) \ e^{-i\Delta k_1 z}
$$

(3.29a)

$$
\partial_z \mathcal{E}_2^+(z, \Omega_2) = -i\kappa_2 \int d\Omega_3 \ \mathcal{E}_1^-(z, \Omega_2 - \Omega_3) \ \mathcal{E}_3^+(z, \Omega_3) \ e^{-i\Delta k_2 z}
$$

(3.29b)

$$
\partial_z \mathcal{E}_3^+(z, \Omega_3) = -i\kappa_3 \int d\Omega_1 \ \mathcal{E}_1^+(z, \Omega_1) \ \mathcal{E}_2^+(z, \Omega_3 - \Omega_1) \ e^{i\Delta k_3 z},
$$

(3.29c)

with

$$
\kappa_\alpha = \frac{\omega_{0\alpha} + \Omega_\alpha}{4\pi cn_\alpha} \chi^{(2)}_{xxy}
$$

$$
\Delta k_1 = k_3(\omega_{03} + \Omega_3) - k_2(\omega_{02} + \Omega_3 - \Omega_1) - k_1(\omega_{01} + \Omega_1)
$$

$$
\Delta k_2 = k_3(\omega_{03} + \Omega_3) - k_2(\omega_{02} + \Omega_2) - k_1(\omega_{01} + \Omega_3 - \Omega_2)
$$

$$
\Delta k_3 = k_3(\omega_{03} + \Omega_3) - k_2(\omega_{02} + \Omega_3 - \Omega_1) - k_1(\omega_{01} + \Omega_1).
$$

The same set of equations is found for type II (eoe) phase matching in negative uniaxial
crystals. Here, the polarizations of the incoming fields are perpendicular to each other
and the generated field is polarized along the extraordinary axis.

In order for the individual integrals to contribute to the result, the differences of the
three wave vectors should also vanish, a condition that is called phase matching and is
related to momentum conservation. As all three interacting fields have a spectral dis-
tribution and thus experience dispersion, phase matching is usually only fulfilled for the
center frequencies. The phase matching condition is sometimes written as a Taylor series,
for example, in the case of the first integral

$$
\Delta k_1 = k_3(\omega_{03} + \Omega_3) - k_2(\omega_{02} + \Omega_3 - \Omega_1) - k_1(\omega_{01} + \Omega_1)
$$

$$
= [k_{03} - k_{02} - k_{01}] + \left[ \frac{dk_3}{d\omega} \right]_{\omega_{03}} \Omega_3 - \frac{dk_2}{d\omega} \right]_{\omega_{02}} (\Omega_3 - \Omega_1) - \frac{dk_1}{d\omega} \right]_{\omega_{01}} \Omega_1 \right] + \cdots
$$
If the lowest-order terms add to zero, the process is labeled phase-matched to first order. If, in addition, the next three terms add to zero, the process is phase-matched to second order, and so on. Note that phase matching to first order is identical to matching the phase velocities at the center frequencies. Phase matching to second order requires matching the group velocities. It has to be emphasized at this point that phase matching is a condition that is fulfilled only if special care is taken. Usually, appropriate crystals have to be employed and, moreover, the crystals have to be aligned correctly with respect to the propagation direction and the polarization of the fields. That is to say, three wave mixing processes are observed to be extremely weak unless the optical setup is deliberately designed. In addition, we have only considered cases where all three fields propagate in the same direction, and consequently, we have to match wave numbers rather than wave vectors. There are situations where this is not the case, and the angles between the wave vectors are nonzero. For example, such geometries are used when both phase velocity matching and group velocity matching must be fulfilled.

A frequently used approximation is the nondepleted fundamental wave approximation. Here, it is assumed that the overall conversion efficiency is low and the two input fields remain constant in amplitude as they propagate through the medium, that is \( E_{1,2}^\pm(z, \Omega) \equiv E_{1,2}^\pm(\Omega) \). Therefore, only one differential equation remains to be solved

\[
\partial_z E_3^+ (z, \Omega_3) = -i \kappa_3 \int d\Omega_1 E_1^+ (\Omega_1) E_2^+ (\Omega_3 - \Omega_1) e^{i\Delta k z}. \tag{3.30}
\]

This equation can be integrated from \( z = 0 \) to \( z = L \) and we find

\[
E_3^+ (L, \Omega_3) = -i \kappa_3 L \int d\Omega_1 E_1^+ (\Omega_1) E_2^+ (\Omega_3 - \Omega_1) \text{sinc} \frac{\Delta kL}{2} e^{i\Delta kl/2}, \tag{3.31}
\]

where \( L \) is the length of the crystal. A special case of SFG is second harmonic generation. Here, the two input fields are identical and we obtain

\[
E_3^+ (L, \Omega_3) = -i \kappa_3 L \int d\Omega_1 E_1^+ (\Omega_1) E_1^+ (\Omega_3 - \Omega_1) \text{sinc} \frac{\Delta kL}{2} e^{i\Delta kl/2}. \tag{3.32}
\]

Figure 3.2 shows the spectral intensity \( |E_3^+ (z, \Omega_3)|^2 \) as the 100-fs-long and 10-fs-long pulses propagate through a suitable nonlinear crystal. The second harmonic field is initially zero and grows through frequency conversion. While the 100-fs-long pulse is frequency doubled almost without any spectral distortions, the 10-fs-long pulse is modified and one can see the effect of the sinc function, that is the effect of phase matching, through the appearance of side bands.
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3.3.1.1 Limit of a Thin Crystal

In the limit of a thin crystal, the phase mismatch $\Delta kL$ is close to zero for all frequency combinations within the spectra of the two input pulses

$$E_3^+(L, \Omega_3) = -i\kappa_3 L \int d\Omega_1 E_1^+(\Omega_1) E_2^+(\Omega_3 - \Omega_1).$$

This situation is desirable for converting broad fundamental spectra, especially in pulse characterization techniques. However, thin crystals necessarily imply a low conversion efficiency and there is always a trade-off between bandwidth and efficiency, which has to be optimized for a given situation.

3.3.1.2 Limit of a Thick Crystal

For very thick crystals, $\Delta kL$ is large except for a single frequency $\Omega_3$. If we assume that the crystal is oriented such that this condition is fulfilled for $\Omega_3 = 0$, then

$$E_3^+(L, 0) = -i\kappa_3 L \int d\Omega_1 E_1^+(\Omega_1) E_2^+(-\Omega_1).$$

If we consider second harmonic generation and split the two input fields into amplitudes and phases, we obtain

$$E_3^+(L, 0) = -i\kappa_3 L \int d\Omega_1 |E_1^+(\Omega_1)| |E_1^+(-\Omega_1)| e^{i\phi(\Omega_1) + i\phi(-\Omega_1)}. $$
Thus, two very distinct situations arise depending on whether the slowly varying phases are odd or even functions with respect to frequency

\[
E_3^+(L, 0) = -i\kappa_3 L \int d\Omega_1 |E_1^+(\Omega_1)| |E_1^+(-\Omega_1)| \left\{ \begin{array}{ll} e^{i2\phi(\Omega_1)} & \text{even} \\ 1 & \text{odd} \end{array} \right.
\]

for odd phase functions, the SHG process is completely independent of the exact nature of the phase modulation. On the contrary, for even phase functions, the detailed shape may even cause the second harmonic signal to completely disappear.

### 3.3.2 Third-Order Nonlinearities: Four Wave Mixing

In materials with inversion symmetry, such as glasses or liquids, all even-order susceptibilities must be zero. Then, third-order nonlinearities become important as they are the lowest-order contribution to the nonlinear material response. For third-order nonlinearities, also labeled four wave mixing processes, the wave equation is almost always solved in the time domain

\[
\frac{1}{c^2} \frac{\partial^2 E(z, t)}{\partial t^2} = \int dt_1 \epsilon(1) E(z, t - t_1) + \chi^{(3)} E(z, t) \cdot E(z, t) \cdot E(z, t).
\]  

(3.33)

Here, the total field may be composed of four individual fields

\[
E(z, t) = \frac{1}{2} \sum_{\alpha=1}^4 \left[ e^{i\omega_0 t - ik_0 z} + \hat{e}_\alpha \cdot E_\alpha(z, t) e^{-i\omega_0 t + ik_0 z} \right].
\]  

(3.34)

Inserting the Ansatz (3.34) in the nonlinear wave equation (3.33) yields a relatively complex expression. With the following approximations, we try to simplify this expression to a level that permits either an analytic solution or leads to a form that can be easily solved by numerical means. First, we replace the linear dispersion term by Fourier transforming it to the frequency domain, expressing the squared wave vector through its Taylor expansion, and Fourier transforming the resulting series expansion back to the time domain. For each of the four fields, we then obtain

\[
\frac{1}{c^2} \frac{\partial^2 E(z, t)}{\partial t^2} \int dt_1 \epsilon(t_1) E_\alpha(z, t - t_1) e^{i\omega_0 (t - t_1) - ik_0 z} = -\sum_{n,m=0}^{\infty} \frac{k_{\alpha n k_{\alpha m}}}{n! m! n+m} \frac{1}{1+n+m} \partial_t^{n+m} E_\alpha(z, t) e^{i\omega_0 t - ik_0 z},
\]  

(3.35)

with \( k_{\alpha n} = \frac{d^n k}{d\omega^n} |_{\omega_0} \). Explicitly evaluating the three lowest-order terms on the right-hand side of Equation 3.35, that is \([n, m] = [0, 0], [0, 1] \) and \([1, 0]\), employing the
slowly varying envelope approximation, writing a separate equation for each field, and multiplying the resulting equation from the left with $\hat{e}_\alpha$ yields

$$
\begin{bmatrix}
\partial_z C_\alpha^+(z, t) + \frac{1}{v_{ga}} \partial_t C_\alpha^+(z, t) + \frac{i}{2k_{0a}} \sum_{n,m=1}^{\infty} \frac{k_{na} k_{ma}}{n! m!} \partial_t^{n+m} E_\alpha^+(z, t) \\
\end{bmatrix} e^{i\omega_{0a} t - i k_{0a} z} = -\frac{i}{8k_{0a} c^2} \sum_{\beta, \gamma, \delta} \hat{e}_\alpha \cdot \chi^{(3)}: \hat{e}_\beta \cdot \hat{e}_\gamma \cdot \hat{e}_\delta \partial_t^2
$$

$$
\times \left\{ C_\beta^+(z, t) C_\gamma^+(z, t) C_\delta^+(z, t) e^{i(\omega_0_\beta + \omega_0_\gamma + \omega_0_\delta) t - i(k_0_\beta + k_0_\gamma + k_0_\delta) z} + C_\beta^+(z, t) C_\gamma^+(z, t) C_\delta^-(z, t) e^{i(\omega_0_\beta + \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta + k_0_\gamma - k_0_\delta) z} + C_\beta^-(z, t) C_\gamma^+(z, t) C_\delta^+(z, t) e^{i(\omega_0_\beta - \omega_0_\gamma + \omega_0_\delta) t - i(k_0_\beta + k_0_\gamma + k_0_\delta) z} + C_\beta^-(z, t) C_\gamma^+(z, t) C_\delta^-(z, t) e^{i(\omega_0_\beta - \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta + k_0_\gamma - k_0_\delta) z} + C_\beta^+(z, t) C_\gamma^-(z, t) C_\delta^+(z, t) e^{i(\omega_0_\beta + \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta + k_0_\gamma - k_0_\delta) z} + C_\beta^-(z, t) C_\gamma^-(z, t) C_\delta^+(z, t) e^{i(\omega_0_\beta - \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta - k_0_\gamma + k_0_\delta) z} + C_\beta^-(z, t) C_\gamma^-(z, t) C_\delta^-(z, t) e^{i(\omega_0_\beta - \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta - k_0_\gamma - k_0_\delta) z} + C_\beta^+(z, t) C_\gamma^-(z, t) C_\delta^-(z, t) e^{i(\omega_0_\beta + \omega_0_\gamma - \omega_0_\delta) t - i(k_0_\beta - k_0_\gamma + k_0_\delta) z} \right\}.
$$

(3.36)

Evaluating the second-order time derivative of each term on the right-hand side of Equation 3.36 yields three terms with decreasing magnitude. Sometimes, it will be sufficient to consider the largest contribution only, as we will see in the next section. Also, we change the coordinate system and transform it to a system that propagates at the group velocity of the field $\alpha$.

3.3.2.1 Third Harmonic Generation

The first practical example of Equation 3.36 is third harmonic generation, where we assume that the three incoming fields are identical and polarized along $x$ and the resulting third harmonic is polarized along $y$. Energy conservation requires $\omega_{0\alpha} = \omega_{0\beta} + \omega_{0\gamma} + \omega_{0\delta}$ and only the first term in Equation 3.36 yields a nonzero contribution, that is

$$
\partial_\xi E_4^+(\xi, \tau) + \frac{i}{2k_{04}} \sum_{n,m=1}^{\infty} \frac{k_{n4} k_{m4}}{n! m!} \partial_\tau^{n+m} E_4^+(\xi, \tau) \approx -\frac{i k_{04} \chi^{(3)}_{yxx}}{8} \varepsilon_{\alpha}^{(3)}(\xi, \tau) e^{i k_{04} z}.
$$

(3.37)
with $\Delta k = k_{04} - 3k_{01}$. The third-order nonlinear susceptibilities are usually low and phase matching is not easy to achieve, that is, efficient third harmonic generation is difficult to realize. This is why third harmonic generation in solid materials has found only a few practical applications.

### 3.3.2.2 Degenerate Four Wave Mixing

If all interacting fields have the same center frequency, the nonlinear interaction is called degenerate four wave mixing (DFWM). It plays an important role in ultrafast optics as it is responsible for an intensity-dependent change of the index of refraction, for self-steepening, it facilitates white light generation, and it is important for soliton formation and pulse propagation in optical fibers. Although beyond the scope of this chapter, it is also responsible for spatial effects, such as self-focusing. The four interacting fields may or may not have the same polarization. If they have the same polarization, then for some terms in Equation 3.36, phase matching is automatically fulfilled, that is for those terms with two positive and two negative wave number contributions. Without the necessity of extra phase matching measures, these processes will be strong in almost any material if only the intensity is high enough.

In the following, we assume that all interacting fields have the same center frequency and are polarized along the same axis. Then, three out of the eight terms on the right-hand side of Equation 3.36 contribute and considering the two largest contributions on the right-hand side of Equation 3.37 yields

$$\partial_\xi \mathcal{E}^+(\xi, \tau) + \frac{i}{2k_0} \sum_{n,m=1}^\infty \frac{k_n k_m}{n! m!} \partial_\tau^{n+m} \mathcal{E}^+(\xi, \tau) \approx -i \gamma \left[ 1 - \frac{2i}{\omega_0} \partial_\tau \right] \mathcal{E}^+(\xi, \tau) \mathcal{E}^+(\xi, \tau)^2$$

(3.38)

where we have used $\gamma = 3\omega_0 \chi^{(3)}_{xxxx} / (8cn_0)$. In the following sections, we will analyze different aspects of this equation. That is, we will assume different approximations with the goal to find analytic solutions for specific situations.

### 3.3.2.3 Self-Phase Modulation

If the dispersion of a material is vanishingly small, the material is very thin, or if the pulses have relatively small spectral bandwidth, we may neglect dispersion and consider only the strongest third-order contribution

$$\partial_\xi \mathcal{E}^+(\xi, \tau) = -i \gamma \mathcal{E}^+(\xi, \tau) \mathcal{E}^+(\xi, \tau)^2.$$  

(3.39)

To solve this equation, we separate the field in its amplitude and phase part and split the resulting differential equation in two, one for the real and one for the imaginary part,
that is

$$\partial_\xi |E^+(\xi, \tau)| = 0 \quad (3.40a)$$
$$\partial_\xi \Phi(\xi, \tau) = -\gamma |E^+(\xi, \tau)|^2. \quad (3.40b)$$

The solution is found to be

$$E^+(\xi, \tau) = |E^+(0, \tau)| e^{-i\gamma|E^+(0, \tau)|^2 \xi}. \quad (3.41)$$

Apparently, the slowly varying amplitude is not changing as the pulse propagates through the medium. The temporal phase, however, grows linearly with $\xi$ and causes a modulation of the spectral amplitude. This is why this process has been named self-phase modulation. Sometimes Equation (3.41) is cast into the following form:

$$E^+(\xi, \tau) = |E^+(0, \tau)| \exp \left[ -i\omega_0/c n_2 I(0, \tau) \xi \right], \quad (3.42)$$

with the nonlinear index of refraction

$$n_2 = \frac{2\gamma}{\varepsilon_0 n_0 \omega_0} = \frac{3\chi_{xxx}}{4\varepsilon_0 c n_0^2}. \quad (3.43)$$

The intensity-induced refractive index change causes an instantaneous phase modulation and the instantaneous frequency can deviate considerably from $\omega_0$, that is

$$\omega(t) - \omega_0 = -\frac{\omega_0}{c} n_2 \partial_\tau I(0, \tau) \xi. \quad (3.44)$$

Figure 3.3 shows the spectrum of a 100-fs-long and a 10-fs-long pulse after undergoing self-phase modulation in a 0.1-mm-thick fused silica plate. Both pulses have the same peak intensity.

What looks like spectral interference is a result of the fact that self-phase modulation produces pairs of instantaneous frequencies at different times. The number of such interference peaks $M$ in the spectrum (see Figure 3.3) increases with intensity and propagation length $\xi$ and is approximately given by

$$M = \frac{1}{2} + \frac{\gamma |E^+(0, 0)|^2 \xi}{\pi} = \frac{1}{2} + \frac{\omega_0 n_2 I_0 \xi}{c \pi}, \quad (3.45)$$

where $I_0$ is the peak intensity of the pulse. Note that self-phase modulation does not always cause spectral broadening. If the pulse has an initial negative chirp, even spectral narrowing may be observed.
3.3.2.4 Two-Photon Absorption

Here, we investigate the case when $\chi^{(3)}$ is not purely real, but has a small imaginary contribution, that is $\chi^{(3)} = \chi^{(3)'} + i\chi^{(3)''}$. Again, we neglect dispersion and the wave equation reads

$$\partial_\xi \mathcal{E}^+ (\xi, \tau) = -i(\gamma' + i\gamma'')\mathcal{E}^+ (\xi, \tau)|\mathcal{E}^+ (\xi, \tau)|^2. \quad (3.46)$$

To find a solution, we separate the field in an amplitude and a phase part, that is $\mathcal{E}^+ (\xi, \tau) = \sqrt{2I(\xi, \tau)/\varepsilon_0cn_0} \exp[i\Phi(\xi, \tau)]$. Inserting this Ansatz in Equation 3.46 yields

$$\partial_\xi I(\xi, \tau) = \frac{8\gamma''}{(\varepsilon_0cn_0)^2} I^2(\xi, \tau) \quad (3.47a)$$

$$\partial_\xi \Phi(\xi, \tau) = -\frac{2\gamma'}{\varepsilon_0cn_0} I(\xi, \tau). \quad (3.47b)$$

The solution is

$$I(\xi, \tau) = \frac{I(0, \tau)}{1 - \frac{4\gamma''}{\varepsilon_0cn_0} I(0, \tau)\xi} \quad (3.48a)$$

$$\Phi(\xi, \tau) = \frac{\gamma'}{2\gamma''} \ln \left[ 1 - \frac{4\gamma''}{\varepsilon_0cn_0} I(0, \tau)\xi \right]. \quad (3.48b)$$

For $\gamma'' < 0$, the intensity decreases as the pulse propagates through the medium and the infinitesimal change is proportional to the square of the intensity; this situation is labeled two-photon absorption.
3.3.2.5 Self-Steepening

Neglecting dispersion but considering the two strongest third-order contributions yields

\[ \partial_\xi \mathcal{E}^+(\xi, \tau) = -i\gamma \left( 1 - \frac{2i}{\omega_0} \right) \mathcal{E}^+(\xi, \tau)|\mathcal{E}^+(\xi, \tau)|^2. \] (3.49)

To find an analytic solution, we separate the field in amplitude and phase, \( \mathcal{E}^+(\xi, \tau) = \sqrt{2} \mathcal{I}(\xi, \tau)/\varepsilon_0 \text{cn} \omega_0 \exp[i\Phi(\xi, \tau)] \) and solve the resulting differential equation for the real and the imaginary part separately, that is

\[ \partial_\xi \mathcal{I}(\xi, \tau) = -\frac{12\gamma}{\varepsilon_0 \text{cn} \omega_0} \mathcal{I}(\xi, \tau) \partial_\tau \mathcal{I}(\xi, \tau) \] (3.50a)

\[ \partial_\xi \Phi(\xi, \tau) = -\frac{2\gamma}{\varepsilon_0 \text{cn} \omega_0} \mathcal{I}(\xi, \tau) - \frac{4\gamma}{\varepsilon_0 \text{cn} \omega_0} \mathcal{I}(\xi, \tau) \partial_\tau \Phi(\xi, \tau). \] (3.50b)

The solution to Equation 3.49a is of the form

\[ \mathcal{I}(\xi, \tau) = \mathcal{I} \left( \tau - \frac{12\gamma}{\varepsilon_0 \text{cn} \omega_0} \xi \right). \] (3.51)

Even in the absence of dispersion, the pulse shape changes slightly upon propagation. It appears as if the group velocity becomes intensity dependent. Those parts of the pulse that have a higher intensity propagate slower than those with a lower intensity and, consequently, the pulse shape will steepen toward the trailing edge. It is because of this behavior that this contribution has been labeled self-steepening.

3.3.2.6 Nonlinear Schrödinger Equation

In the following, we allow for dispersion but consider only second-order dispersion. In this case, the so-called nonlinear Schrödinger equation results

\[ \partial_\xi \mathcal{E}^+(\xi, \tau) - \frac{ik_2}{2} \partial_\tau^2 \mathcal{E}^+(\xi, \tau) = -i\gamma \mathcal{E}^+(\xi, \tau)|\mathcal{E}^+(\xi, \tau)|^2. \] (3.52)

With a change of variables, that is \( \zeta = \xi |k_2|/\Delta t_o^2 \) and \( \Theta = \tau / \Delta t_0 \), where \( \Delta t_0 \) is some pulse duration, and the substitution \( u(\zeta, \Theta) = \sqrt{\Delta t_0^2 \gamma / |k_2|} \mathcal{E}^+(\xi, \tau) \), we find

\[ i\partial_{\zeta} u(\zeta, \Theta) - \frac{1}{2} \partial_\Theta^2 u(\zeta, \Theta) - u(\zeta, \Theta) |u(\zeta, \Theta)|^2 = 0, \] (3.53)

given that \( \gamma > 0 \) and \( k_2 < 0 \). This equation has the solution

\[ u(\zeta, \Theta) = \text{sech} \Theta \, e^{-i\zeta/2}, \] (3.54)
as can be easily shown. That is, the solution is a pulse with a temporal amplitude invariant with propagation. Such a pulse is called an optical soliton [5]. In terms of the original variables, we find

$$E^+(\xi, \tau) = \sqrt{\frac{|k_2|}{\gamma \Delta t_0^2}} \frac{1}{\text{sech} \left( \frac{\tau}{\Delta t_0} \right)} \exp \left( -i \frac{|k_2|}{2 \Delta t_0^2} \right).$$ (3.55)

Inspecting this solution shows that an optical soliton is only found if a number of requirements are fulfilled. First, the medium must have a positive nonlinear constant and a negative GVD (or vice versa). Second, given the material parameters, the amplitude and the temporal width of the soliton are linked to each other and cannot be chosen arbitrarily. Third, the soliton acquires no temporal phase upon propagation, that is, not only the temporal but also the spectral amplitude remains unaltered. Besides the fundamental soliton, there exist higher-order solitons of order \(N\) whenever the amplitude satisfies \(N \sqrt{|k_2|/(\gamma \Delta t_0^2)}\), with \(N = 1, 2, \ldots\) Figure 3.4 shows the temporal and spectral intensity of a pulse propagating through a suitable medium. If the nonlinearity is set to zero (top row), the pulse undergoes dispersion and as a consequence broadens. If the dispersion is set to zero but the nonlinearity is nonzero, the spectrum broadens and shows the characteristic sequence of peaks. If both dispersion and nonlinearity are nonzero, neither the temporal nor the spectral shape of the pulse seems to change upon propagation, indicating the existence of a soliton.

3.3.2.7 Intrapulse Raman Scattering

In order to formulate an appropriate equation describing the propagation of pulses through very long media, such as optical fibers, we need to include the Raman effect. Laser pulses may couple to molecular vibrations or optical phonons, which typically have much lower eigenfrequencies, through an induced polarizability. To gain some insight, we again treat the system classically and assume that the material is composed of independent oscillators. The driving force is proportional to the derivative of the time-averaged electric energy density with respect to the direction of motion \(s\), that is

$$F(z, t) = \frac{1}{2} \varepsilon_0 \partial_s \alpha |_{s=0} \langle E^2(z, t) \rangle,$$ (3.56)

with the polarizability \(\alpha\). The time average reflects the fact that the resonance frequencies of molecules or phonons are much lower than the center frequency of the laser pulse. The solution may be found in the frequency domain

$$s(z, \omega) = \frac{\varepsilon_0}{2m} \partial_s \alpha |_{s=0} \frac{1}{\omega_0^2 - \omega^2 + i \gamma \omega} \int dt \langle E^2(z, t) \rangle e^{-i \omega t}.$$ (3.57)
The macroscopic polarization is $P(z,t) = N\alpha(z,t)E(z,t)$ and we are only interested in the field-induced part, that is $P(z,t) = N\delta_{ij}\alpha(z,t)|_{\delta=0} s(z,t)\overline{E}(z,t)$. With this and using $\langle E^2(z,t) \rangle = |\mathcal{E}^+(z,t)|^2/2$, we find for the slowly varying polarization

$$\mathcal{P}^+(z,t) = \varepsilon_0 \int dt\bar{R}(t_1)|\mathcal{E}^+(z,t-t_1)|^2 \mathcal{E}^+(z,t), \quad (3.58)$$
with $R_R(t)$ being the Fourier transform of

$$
\frac{N}{2m} \left( \partial_s \omega \right) \left|_{\omega = \omega_0} \right)^2 \frac{1}{\omega_0^2 - \omega^2 + i\gamma \omega}.
$$

(3.59)

Inspecting the imaginary part of the Raman susceptibility indicates absorption for positive relative frequencies and gain for negative relative frequencies. That is, the Raman effect shifts pulses toward lower frequencies as they propagate through the medium.

**FIGURE 3.5** Temporal and spectral intensity as a function of propagation distance through a 100-mm-long optical fiber. The center wavelength of the pulse is in the normal dispersion region (top), coincides with the zero dispersion wavelength (center), or is within the anomalous dispersion region (bottom).
3.3.2.8 Optical Fibers

Adding the Raman contribution to Equation 3.38 results in an equation that is commonly used to describe the propagation of time domain pulses in optical fibers [6]

\[
\frac{\partial \xi}{\partial \tau} E^+(\xi, \tau) + i \sum_{n,m=1}^{\infty} \frac{k_n k_m}{n! m!} i^{n+m} \partial_{\tau}^{n+m} E^+(\xi, \tau) \\
= -i\gamma \left[ 1 - \frac{2i}{\omega_0} \frac{\partial \tau}{\partial \xi} \right] E^+(\xi, \tau) \int d\tau \, R(\tau_1) |E^+(\xi, \tau - \tau_1)|^2,
\]

(3.60)

where we have combined the instantaneous electronic response \(\delta(t)\) and the Raman response \(R_R(t)\) of the medium to \(R(t) = (1 - f) \delta(t) + f R_R(t)\). The relative weight is governed by the constant \(f\), which for fused silica is approximately 0.18.

Figure 3.5 shows the evolution of the temporal and the spectral intensity as a function of propagation distance through a 100-mm-long fiber. The zero GVD wavelength of the optical fiber is around 754 nm. If we inject a 20-fs-long laser pulse with a center wavelength of 654 nm, we are well within the normal dispersion regime and we observe spectral broadening due to self-phase modulation and substantial broadening due to dispersion (top row). After a few millimeters, the pulse broadens so much that the intensity drops below a level where it can cause self-phase modulation and the spectral broadening stops. Conversely, if the center wavelength is well within the anomalous dispersion regime, the temporal intensity profile shows the formation of several solitons, which due to their different spectral content propagate with different group velocities (bottom row). The spectrum is structured, also because of interference effects. Injecting a pulse with a center wavelength that matches the zero dispersion wavelength shows a mixture of the two scenarios discussed (center row).

3.4 Time Domain Pulses in Linear Optical Systems

In practically every ultrafast experiment, we find optical units that are designed to change the properties of a time domain pulse in a very specific way. Such systems can often be described by means of a linear transfer function thereby neglecting the detailed layout of the unit. They manipulate the spectral amplitude \(|E_{in}^+(\omega)|\) or phase \(\phi(\omega)\) of the incident pulse through a linear operation, that is

\[
E_{out}^+(\omega) = E_{in}^+(\omega) M(\omega) = |E_{in}^+(\omega)| |M(\omega)| e^{i\phi(\omega) + i\Phi(\omega)}.
\]

(3.61)

If \(|M(\omega)| \neq 1\), we find amplitude modulation, and if \(\Phi(\omega) \neq 0\), phase modulation. In the following section, we start by introducing systems that are based on angular dispersion and mostly aim at stretching and compressing pulses in a well-controlled way. The next class of systems is based on resonances and generally modifies both amplitude and phase of the incident pulse. We conclude this section with the most flexible device,
a so-called pulse shaper, which can manipulate the spectral amplitude and phase in an almost arbitrary way. More details can be found in a number of textbooks [7].

### 3.4.1 Systems Based on Angular Dispersion

The phase modulation in optical systems based on angular dispersion is mostly a result of the fact that different colors within the spectrum travel along different paths. Loosely speaking, the phase is frequency dependent because the geometrical path length is a function of the frequency. In the following, we neglect all amplitude modulations that may arise due to spectral clipping, wavelength-dependent reflectivities at interfaces, and so on. Furthermore, at the exit of the optical system, the pulse must be recovered, that is, all angular dispersion present at some point within the system must be compensated for by the time the pulse has passed through the entire system. Our task is to find the phase modulation for a given sequence of optical elements, that is, the optical path length as a function of frequency. There are different routes to proceed from here.

The first is based on a general argument, which is valid for any system possessing angular dispersion. Suppose an incoming pulse impinging on a dispersive element after which the spectrum is dispersed and different colors travel in different directions. We select two diffracted rays, namely, the reference ray, which corresponds to the center frequency, and any other ray. The distance between the dispersive element and the plane of interest measured along the reference ray is $L$. It is easy to see that the phase front of the arbitrary ray intersects with the phase front of the reference ray at the plane of interest, if the length of the corresponding ray is $L \cos \Delta \beta$, where $\Delta \beta = \beta(\omega) - \beta_0$ is the relative diffraction angle. With this, the phase modulation becomes

$$\Phi(\omega) = \frac{\omega}{c} L \cos \Delta \beta. \quad (3.62)$$

Equation 3.62 produces correct results if no dispersive material is involved. From Equation 3.62, analytic expressions for the Taylor coefficients of the phase expansion can be obtained. The quadratic phase coefficient amounts to

$$\Phi_2 = -\frac{L\omega_0}{c} \left( \frac{d\omega \beta}{|\omega_0|} \right)^2, \quad (3.63)$$

where we have assumed that $\Delta \beta \ll 1$. Note that Equation 3.63 is always negative irrespective of the sign of the angular dispersion. Similarly, we find for the cubic phase

$$\Phi_3 = -\frac{3L}{c} \left[ \left( \frac{d\omega \beta}{|\omega_0|} \right)^2 + \omega_0 \frac{d\omega \beta}{|\omega_0|} \frac{d^2\omega \beta}{|\omega_0|} \right], \quad (3.64)$$

which may be positive or negative depending on the angular dispersion.

A different approach to calculate the total phase modulation of an optical system is to treat the system as a sequence of homogeneous materials separated by interfaces with
specific properties. The total phase then is

$$\Phi(\omega) = \sum_j \left[ \frac{\omega}{c} P_j(\omega) + R_j(\omega) \right].$$

(3.65)

The optical path length $P_j(\omega) = n_j(\omega)z_j(\omega)$ is the product of the refractive index $n_j(\omega)$ and the frequency-dependent geometrical path length $z_j(\omega)$ between two interfaces. At the interfaces, an additional phase jump $R_j(\omega)$ may have to be considered.

The last approach uses the fact that the first derivative of the spectral phase with respect to frequency can be interpreted as the “time” each frequency requires to travel through the optical system

$$t(\omega) = \frac{d}{d\omega} \Phi(\omega).$$

(3.66)

Brorson and Haus have shown that for an optical system without dispersive material

$$t(\omega) = \frac{P(\omega)}{c}$$

(3.67)

is a consequence of Fermat’s principle. Therefore, if the phase modulation of the complete optical system is written as

$$\Phi(\omega) = \frac{\omega}{c} P(\omega) + R(\omega),$$

(3.68)

we find with Equations 3.66 and 3.67 by comparison that

$$d_\omega R(\omega) = -\frac{\omega}{c} d_\omega P(\omega) \quad \text{and} \quad R(\omega) = -\frac{1}{c} \int d\omega' \omega' d_\omega' P(\omega').$$

(3.69)

Next, we will apply the different approaches to three relatively simple, but widely used, systems, namely, the grating compressor, the grating stretcher, and the prism compressor.

### 3.4.1.1 Grating Compressor

Figure 3.6 shows a grating compressor consisting of two parallel gratings and an end mirror. The incoming beam is spectrally dispersed on the first grating, impinges onto the second grating, and is sent back through the system by the end mirror. To calculate the optical path $P(\omega)$ as a function of frequency, the full path is split into two parts. The first part is the distance between the two gratings $AB = L/\cos \beta$, and the second part is the distance between the second grating and the mirror (see Figure 3.6) $BC = r - B_0B \cos(90 - \alpha)$, where $r$ is the distance between the second grating and the end mirror. The total path is the sum of the two contributions

$$ABC = \frac{L}{\cos \beta} + r - L \tan \beta_0 \sin \alpha + L \tan \beta \sin \alpha,$$

(3.70)
where $\beta_0$ is the diffraction angle at center frequency. From Equation 3.70, we calculate $P(\omega)$, $R(\omega)$, and the total phase modulation $\Phi(\omega)$. Note that in a full compressor the pulse travels to the end mirror and back again through the whole system, yielding

$$\Phi(\omega) = \frac{2\omega}{c} \left[ \frac{L}{\cos \beta} + r + L(\tan \beta - \tan \beta_0) \sin \alpha \right] - \frac{4\pi L}{G} \tan \beta. \quad (3.71)$$

From the phase, we derive the second- and third-order Taylor coefficients

$$\Phi_2 = -\frac{8\pi^2 c L}{G^2 \omega_0^3} \frac{1}{\cos^3 \beta_0} = -\frac{\lambda_0}{\pi c^2} \left( \frac{\lambda_0}{G} \right)^2 \frac{L}{\cos^3 \beta_0}, \quad (3.72)$$

$$\Phi_3 = \frac{3\lambda_0^2}{2\pi^2 c^3} \left( \frac{\lambda_0}{G} \right)^2 \frac{L}{\cos^3 \beta_0} \left[ 1 + \frac{\lambda_0}{G} \frac{\sin \beta_0}{\cos^2 \beta_0} \right]. \quad (3.73)$$

The characteristic length is the normal distance between the two gratings $L$. Table 3.1 lists the second- and third-order dispersion coefficients for different center wavelengths and grating constants. The angle of incidence is arbitrarily fixed to $30^\circ$. Note that the second- and third-order dispersions of a grating compressor typically have opposite signs.

### 3.4.1.2 Grating Stretcher

As we have seen in the previous section, a grating compressor consists of only two gratings and has a negative second-order dispersion. Grating stretchers are used to compensate for this negative dispersion or, to put it in the right perspective, to generate a positive dispersion in the first place. Obviously, these devices require more than just two gratings. Let us start with a classic stretcher design, as shown in Figure 3.7. The incident beam is dispersed on the first grating, passes through a series of two identical lenses with focal length $f$, and is recombined at the second grating. Then it is sent all the way back through the optical system by the end mirror. If all optical elements are separated by one
TABLE 3.1  Second- and Third-Order Dispersion for a Grating Compressor with Different Gratings

<table>
<thead>
<tr>
<th>λc (nm)</th>
<th>Lines/mm</th>
<th>Φ2/L (fs²/cm)</th>
<th>Φ3/L (fs³/cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>600</td>
<td>−882</td>
<td>527</td>
</tr>
<tr>
<td>400</td>
<td>1200</td>
<td>−3292</td>
<td>2086</td>
</tr>
<tr>
<td>400</td>
<td>1800</td>
<td>−7727</td>
<td>4603</td>
</tr>
<tr>
<td>800</td>
<td>300</td>
<td>−1754</td>
<td>2091</td>
</tr>
<tr>
<td>800</td>
<td>600</td>
<td>−6545</td>
<td>8290</td>
</tr>
<tr>
<td>800</td>
<td>1200</td>
<td>−32,723</td>
<td>32,140</td>
</tr>
<tr>
<td>1064</td>
<td>300</td>
<td>−3971</td>
<td>6372</td>
</tr>
<tr>
<td>1064</td>
<td>600</td>
<td>−15,663</td>
<td>29,000</td>
</tr>
<tr>
<td>1064</td>
<td>1200</td>
<td>−153,724</td>
<td>520,604</td>
</tr>
</tbody>
</table>

Note: First column: wavelength; second column: grating constant; third column: second-order dispersion; and fourth column: third-order dispersion. The angle of incidence on the first grating in all cases is 30°.

focal length, the net phase modulation is zero (4f setup). This configuration is labeled “zero dispersion compressor” (Figure 3.7a).

If both gratings are translated simultaneously toward the lenses by a distance $g$, the optical setup shows positive dispersion. If the gratings are moved further away, the dispersion becomes negative. What is commonly labeled a “stretcher” is actually able to act as a stretcher or a compressor depending on the position of the gratings with respect to the lenses. Chromatic dispersion due to the two lenses can be avoided if the lenses are replaced by reflective optics, as shown in Figure 3.7d. The number of elements may be further reduced if the setup is folded along the line of symmetry. Figure 3.7e displays the grating stretcher geometry that is used in most standard laser systems. We now calculate the optical path of each spectral component in order to obtain a quantitative measure of the phase modulation introduced. With all beams propagating in vacuum, $P(\omega)$, $R(\omega)$,
and, therefore, $\Phi(\omega)$ are readily obtained:

$$
\Phi(\omega) = \frac{\omega}{c} \left[ 8f + 2r - 4g \frac{\cos \beta_0 - \sin(\beta_0 - \beta) \sin \alpha}{\cos \beta} \right] + \frac{8\pi g \cos \beta_0}{G} \tan \beta.
$$

(3.74)

Before comparing the results to those of the grating compressor, we calculate the expressions for second- and third-order dispersion:

$$
\Phi_2 = \frac{\lambda_0}{\pi c^2} \left( \frac{\lambda_0}{G} \right)^2 \frac{2g}{\cos^2 \beta_0}
$$

(3.75)

$$
\Phi_3 = -\frac{3\lambda_0^2}{2\pi^2 c^3} \left( \frac{\lambda_0}{G} \right)^2 \frac{2g}{\cos^2 \beta_0} \left[ 1 + \frac{\lambda_0}{G} \sin \beta_0 \cos^2 \beta_0 \right].
$$

(3.76)

Clearly, the results are identical to Equations 3.72 and 3.73 if we replace $2g \cos \beta_0$ by $(-L)$. Therefore, it is possible to design a stretcher–compressor pair with opposite phase modulation properties by adjusting $L$, $g$, and $\beta_c$ accordingly. This is at the heart of all chirped pulse amplification (CPA) schemes, which stretch the pulse prior to amplification and then compress it again and rely on the fact that stretching and compressing preserves the temporal shape of the input pulse as much as possible.

**FIGURE 3.8** (a) Geometry of a prism compressor consisting of two prisms and an end mirror. The pulse enters the prism sequence from the left, is reflected at the end mirror, and passes the prism sequence a second time. (b) Beam path from the tip of the first prism (point C) through the second prism for the center frequency (solid curve) and some other frequency (dashed curve).
3.4.1.3 Prism Compressor

The geometry of a typical prism compressor is depicted in Figure 3.8a. Two identical isosceles prisms are arranged opposite to each other with their inner sides parallel. The laser pulse comes from the left, is dispersed by the first prism, and propagates to the second prism, after which all frequencies are parallel again. The end mirror reverses the direction and the pulse travels back through the entire system. The prism geometry, that is the apex angle, and the angle of incidence $\alpha_1$ are usually, but not necessarily, determined by two constrains. First, in order to minimize reflection losses, the electric field is p-polarized with respect to the prism faces and the angle of incidence is identical to Brewster’s angle. Second, the apex angle is such that the center frequency experiences minimum deviation, that is it travels through the prism parallel to its base to avoid any astigmatism to the beam. In order to derive analytic expressions for the second- and the third-order phase, it is instructive to evaluate the situation depicted in Figure 3.8b, which shows the second prism only. We assume that the pulse is refracted at the tip of the first prism (C in Figure 3.8b). Then possible wave fronts at center frequency are $AC$ and $BE$.

From this, we deduce that $CDE = AB$ and we may apply the formalism derived in the beginning of this chapter with $L$ being the distance between the two prism tips. With the angle of incidence being equal to Brewster’s angle and taking into account that the pulse travels twice through the two prisms, one finds

$$\Phi_2 = -\frac{4\lambda_0^3 L}{\pi c^2} \left( d_\lambda n|_{\lambda_0} \right)^2$$

(3.77)

$$\Phi_3 = -\frac{6\lambda_0^4 L}{\pi^2 c^3} \left\{ \left( d_\lambda n|_{\lambda_0} \right)^2 \left[ 1 - \lambda_0 d_\lambda n|_{\lambda_0} \left( \frac{1}{n^3} - n^2 \right) \right] 
+ \lambda_0 \left( d_\lambda n|_{\lambda_0} d_\lambda^2 n|_{\lambda_0} \right) \right\}.$$  (3.78)

As stated before, these equations do not account for the chromatic dispersion introduced by the two prisms themselves. That is why the second-order dispersion is negative as soon as the distance $L$ is greater than zero, which in practice is not the case. The dashed curve in Figure 3.9 shows $\Phi_2$ as a function of prism separation as calculated from Equation 3.77. Taking into account the material dispersion of the prisms themselves shows that for zero prism separation the second-order phase is positive and the prisms need to have a minimal separation before $\Phi_2$ becomes negative.

3.4.2 Systems Based on Resonances

In optical systems, such as etalons, interferometers, or multilayer systems, the output pulse may be interpreted as a result of interferences between an infinite number of time-delayed replicas. Interferences lead to resonant behavior and it is expected that such systems display a complex modulation behavior, which generally cannot be described by a pure phase modulation. We start with the most general case, namely, a sequence of homogeneous media separated by parallel plane interfaces. From the general case, we derive, as an example, the transfer function of a Fabry–Perot etalon. After this, we

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introduce the Fabry–Perot interferometer and a close relative, the Gires–Tournois interferometer. Note that some of these devices may have two rather than one output channels, for example, the Fabry–Perot interferometer has a reflected and a transmitted pulse, which are calculated from the incident pulse by applying two different transfer functions.

3.4.2.1 Multilayer System

Multilayer systems are composed of a stack of layers with different relative permittivities and thicknesses and may be considered as a piecewise homogeneous medium. All interfaces are perpendicular to the z-axis and without any loss of generality we assume that the wave vector is within the $yz$ plane. In optics, the angle of incidence is usually the angle between the wave vector and the normal to the interface. If the incidence angle is nonzero, two cases need to be distinguished. The electric field vector is either perpendicular or parallel to the $yz$ plane. While the first case is referred to as s- or TE-polarization, the latter case is called p- or TM-polarization. Any other state of polarization can be projected into the s/p basis system. From electromagnetic field theory, we know that all tangential field components plus the tangential wave vector component are preserved when passing from one medium to the next. With this knowledge, it is possible to find the transfer functions of a multilayer system for the two states of polarization. In the following, we will use the notion $M_{ab}(\omega)$ with $a = r,t$ for reflection or transmission, and, $b = s,p$ for TE(s) or TM(p) polarization. Thus, a multilayer system in general has four transfer functions. A multilayer system can be designed to have almost any desired reflection or transmission as a function of wavelength. Limitations are mostly technological in nature, for example, the maximum number of layers is currently restricted to a few hundred. Often, multilayer systems are designed to be high reflectivity mirrors, beam splitters, or harmonic separators to name but a few. It should be mentioned that an ideal mirror for beam steering must have a vanishingly small phase modulation as any phase distortion has to be
avoided. There are, however, cases where mirrors are used to introduce or compensate for a specific phase modulation. Those mirrors are commonly referred to as chirped mirrors.

### 3.4.2.1 TE (s) Polarization

For TE polarization, the only nonzero component of the electric field is \( E_x(y, z, \omega) \) and the magnetic field has components parallel to \( y \) and \( z \). Boundary conditions require \( E_x, H_y, \) and \( k_z \) to be continuous. With the Ansatz \( E_x(y, z, \omega) = E_x(y, \omega) \exp(-ik_zz) \), we find with the help of Maxwell’s equations and the Helmholtz equation

\[
\left( \frac{\partial^2}{\partial z^2} + k_{jz}^2 \right) E_x(z, \omega) = 0
\]

(3.79a)

\[
H_y(z, \omega) = \frac{i}{\omega \mu_0} \frac{\partial}{\partial z} E_x(z, \omega)
\]

(3.79b)

\[
H_z(z, \omega) = -\frac{k_z}{\omega \mu_0} E_x(z, \omega),
\]

(3.79c)

with \( k_{jz} \) being the \( z \) component of the wave vector in layer \( j \). After some algebra, it can be shown that the continuous fields at the exit side of each layer are obtained from the fields at the input side of each layer through

\[
\left( \begin{array}{c} E_x(d_j, \omega) \\ i\omega \mu_0 H_y(d_j, \omega) \end{array} \right) = m_{j}^{\text{TE}}(d_j) \left( \begin{array}{c} E_x(0, \omega) \\ i\omega \mu_0 H_y(0, \omega) \end{array} \right),
\]

(3.80)

with \( d_j \) being the thickness of layer \( j \) and the matrix \( m_{j}^{\text{TE}}(d_j) \) is found to be

\[
m_{j}^{\text{TE}}(d_j) = \begin{pmatrix} \cos(k_{jz}d_j) & -\sin(k_{jz}d_j)/k_{jz} \\ k_{jz} \sin(k_{jz}d_j) & \cos(k_{jz}d_j) \end{pmatrix},
\]

(3.81)

where \( k_{jz} = \sqrt{k_j^2 - k_z^2} \). For \( N \) subsequent layers, the matrix of the total system is obtained through successive matrix multiplication

\[
M^{\text{TE}} = \prod_{j=1}^{N} m_{j}^{\text{TE}}(d_j).
\]

(3.82)

From the boundary conditions at the first and the last layer, we obtain the reflection and transmission coefficients of the whole system:

\[
M_{rs}(\omega) = \frac{(k_{lz}M_{22} - k_{Tz}M_{11}) - i(M_{21} + k_{lz}k_{Tz}M_{12})}{(k_{lz}M_{22} + k_{Tz}M_{11}) + i(M_{21} - k_{lz}k_{Tz}M_{12})},
\]

(3.83)

\[
M_{ts}(\omega) = \frac{2k_{lz}}{(k_{lz}M_{22} + k_{Tz}M_{11}) + i(M_{21} - k_{lz}k_{Tz}M_{12})},
\]

(3.84)
where the $M_{ij}$ are the four matrix elements of $M^{\text{TE}}$. The $z$ components of the incident and
the transmitted wave vector are derived from the angle of incidence through

$$k_Iz = \frac{\omega}{c} n_I \cos \theta_I \quad (3.85a)$$

$$k_Tz = \frac{\omega}{c} \sqrt{n_T^2 - n_I^2 \sin^2 \theta_I} \quad (3.85b)$$

3.4.2.1.2 TM (p) Polarization

For TM polarization, the magnetic field is parallel to the $x$-direction and the electric
field has components along $y$ and $z$. Boundary conditions require $H_x$, $E_y$, and $k_y$ to be
continuous. Similarly to the previous section, we find

$$M_{\text{rp}}(\omega) = \frac{(\epsilon_T k_Iz - \epsilon_I k_Tz M_{12}) - i(\epsilon_I \epsilon_T M_{21} + k_Iz k_Tz M_{12})}{(\epsilon_T k_Iz + \epsilon_I k_Tz M_{11}) + i(\epsilon_I \epsilon_T M_{21} - k_Iz k_Tz M_{12})} \quad (3.86)$$

$$M_{\text{tp}}(\omega) = \sqrt{\frac{\epsilon_I}{\epsilon_T}} \frac{2\epsilon_T k_Iz}{(\epsilon_T k_Iz M_{22} + \epsilon_I k_Tz M_{11}) + i(\epsilon_I \epsilon_T M_{21} - k_Iz k_Tz M_{12})} \quad (3.87)$$

with $\epsilon_I$ being the relative permittivity in the first layer and $\epsilon_T$ the relative permittivity of
the last layer, respectively.

3.4.2.2 Fresnel Equations

First, we apply the formalism to the most simple system, namely, a single interface in
which case we have

$$M^{\text{TE}} = M^{\text{TM}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.88)$$

With the help of Equations 3.83 and 3.84, we find for the TE polarized light

$$M_{ts}(\omega) = \frac{k_Iz - k_Tz}{k_Iz + k_Tz} = \frac{n_I \cos \theta_I - n_T \cos \theta_T}{n_I \cos \theta_I + n_T \cos \theta_T} \quad (3.89)$$

$$M_{ts}(\omega) = \frac{2k_Iz}{k_Iz + k_Tz} = \frac{2n_I \cos \theta_I}{n_I \cos \theta_I + n_T \cos \theta_T} \quad (3.90)$$

and with Equations 3.86 and 3.87

$$M_{\text{tp}}(\omega) = \frac{\epsilon_T k_Iz - \epsilon_I k_Tz}{\epsilon_T k_Iz + \epsilon_I k_Tz} = \frac{n_T \cos \theta_T - n_I \cos \theta_T}{n_T \cos \theta_T + n_I \cos \theta_T} \quad (3.91)$$

$$M_{\text{tp}}(\omega) = \sqrt{\frac{\epsilon_T}{\epsilon_I}} \frac{2\epsilon_I k_Iz}{\epsilon_T k_Iz + \epsilon_I k_Tz} = \frac{n_I}{n_T} \frac{2n_I \cos \theta_I}{n_T \cos \theta_T + n_I \cos \theta_T}. \quad (3.92)$$

Equations 3.89 through 3.92 are known as Fresnel equations for a beam passing from
a medium with a refractive index of $n_I$ to a medium with a refractive index of $n_T$. 

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3.4.2.3 Fabry–Perot Etalon

The next example is that of a Fabry–Perot etalon. It is composed of a medium (2) with thickness $d$ sandwiched in between air (1), that is, we have two interfaces. Its characteristic matrices for TE(s) and TM(p) polarization are

$$M^{\text{TE}}(d) = \begin{pmatrix} \cos(k_{1z}d) & -\sin(k_{1z}d)/k_{1z} \\ k_{1z} \sin(k_{1z}d) & \cos(k_{1z}d) \end{pmatrix}$$

(3.93)

and

$$M^{\text{TM}}(d) = \begin{pmatrix} \cos(k_{1z}d) & -\epsilon_{1}/k_{1z} \sin(k_{1z}d) \\ k_{1z} \sin(k_{1z}d) & \cos(k_{1z}d) \end{pmatrix}.$$  

(3.94)

The incident wave vector equals the transmitted wave vector, that is $k_{IZ} = k_{TZ}$ and with this and Equations 3.83 and 3.84, we find for the TE polarized light

$$M_{rst}(\omega) = \frac{-i \sin \delta \left( \frac{k_{1z}}{k_{IZ}} - \frac{k_{IZ}}{k_{1z}} \right)}{2 \cos \delta + i \sin \delta \left( \frac{k_{1z}}{k_{IZ}} + \frac{k_{IZ}}{k_{1z}} \right)},$$

(3.95)

$$M_{ts}(\omega) = \frac{2}{2 \cos \delta + i \sin \delta \left( \frac{k_{1z}}{k_{IZ}} + \frac{k_{IZ}}{k_{1z}} \right)},$$

(3.96)

where we have used $\delta \equiv k_{1z}d$. With Equations 3.86 and 3.87, one obtains

$$M_{rp}(\omega) = \frac{-i \sin \delta \left( \frac{\epsilon_{1}k_{1z}}{\epsilon_{1}k_{IZ}} - \frac{\epsilon_{1}k_{IZ}}{\epsilon_{1}k_{1z}} \right)}{2 \cos \delta + i \sin \delta \left( \frac{\epsilon_{1}k_{1z}}{\epsilon_{1}k_{IZ}} + \frac{\epsilon_{1}k_{IZ}}{\epsilon_{1}k_{1z}} \right)},$$

(3.97)

$$M_{tp}(\omega) = \frac{2}{2 \cos \delta + i \sin \delta \left( \frac{\epsilon_{1}k_{1z}}{\epsilon_{1}k_{IZ}} + \frac{\epsilon_{1}k_{IZ}}{\epsilon_{1}k_{1z}} \right)}.$$  

(3.98)

Figure 3.10 shows the two transfer functions for a Fabry–Perot etalon, which is a 10-μm-thick fused silica plate in air, and the angle of incidence is 30°. The etalon introduces a wavelength-dependent loss to the transmitted beam. By rotating the plate, the positions of maximum loss can be fine-tuned to different wavelengths.

3.4.2.4 Fabry–Perot Interferometer

A Fabry–Perot interferometer is almost identical to a Fabry–Perot etalon; however, the reflectivity at the two interfaces is dominated by separate coatings (metallic or dielectric) rather than by Fresnel equations. The medium has a thickness of $d$ and is sandwiched in between two partially reflective mirrors. The two interfaces, 1 and 2, are not necessarily identical and may have different reflectivities. The multilayer approach is somewhat
impractical to apply here, because of the different types of mirrors that could potentially be used. That is why one usually refers to an alternative method, which is based on accounting for all possible transmission or reflection pathways. The reflected and the transmitted fields are obtained as

\[
M_r(\omega) = \frac{r_{12} + (t_{11}t_{12} - r_{11}r_{12})r_{22} e^{-2i kl}}{1 - r_{11}r_{22} e^{-2i kl}}, \quad (3.99)
\]

\[
M_t(\omega) = \frac{t_{12}r_{22} e^{-ikL}}{1 - r_{11}r_{22} e^{-2i kl}}. \quad (3.100)
\]

To proceed, we concentrate on a single interface. By reversing the beam path of the reflected and the transmitted beam and sending the complex conjugate of both through the interface in reverse direction, we find the following relation:

\[
t_{11}t_{12} - r_{11}r_{12} = \frac{t_{12}}{r_{12}^*} = -\frac{r_{12}}{r_{11}^*}. \quad (3.101)
\]

With this relation, we obtain for the reflected part

\[
M_r(\omega) = \frac{r_{12} - \frac{r_{12}^*}{r_{11}} r_{22} e^{-2i kl}}{1 - r_{11}r_{22} e^{-2i kl}}. \quad (3.102)
\]

If we assume that the interferometer has dielectric mirrors on both sides, for which

\[
\arg(r_{11}) = \pi \quad \arg(r_{21}) = 0 \\
\arg(r_{12}) = 0 \quad \arg(r_{22}) = \pi
\]
\[ |r_{11}| = |r_{12}| \quad |r_{21}| = |r_{22}| \]
\[ \Rightarrow r_{11} = -r_{12} = r_{11}^* \]

is approximately true, and if we further assume that the interferometer is symmetric, that is both mirrors are identical, we may derive

\[ M_r(\omega) = \frac{\sqrt{R} \left( 1 - e^{-2ikL} \right)}{1 - R e^{-2ikL}} \] (3.103)
\[ M_t(\omega) = \frac{(1 - R) e^{-ikL}}{1 - R e^{-2ikL}}, \] (3.104)

with \( r_{12} = \sqrt{R} \) and \( t_{12} = \sqrt{1 - R} \). Of course \( R \) may still be dependent on the polarization if the angle of incidence is nonzero. Generally, such interferometers are operated in two different regimes. First, if the round trip time through the interferometer is larger than the pulse duration, the Fabry–Perot interferometer will produce a train of pulses with decreasing amplitude. Second, if the round trip time is smaller than the pulse duration, the pulse will appear stretched in time and thus the interferometer may be used as a stretcher or a compressor.

### 3.4.2.5 Gires–Tournois Interferometer

The Gires–Tournois interferometer is a special case of the Fabry–Perot interferometer with one of the two mirrors being 100% reflective, that is \(|r_{22}| = 1 \Rightarrow r_{22} = -1\) and \(t_{22} = 0\). Thus, there is no transmitted beam and the amplitude of the reflected beam must be equal to one and

\[ M_r(\omega) = \frac{\sqrt{R} - e^{-2ikL}}{1 - \sqrt{R} e^{-2ikL}}. \] (3.105)

The Gires–Tournois interferometer introduces a pure phase modulation

\[ \Phi(\omega) = \arctan \left[ \frac{(1 - R) \sin(2kL)}{2\sqrt{R} - (1 + R) \cos(2kL)} \right]. \] (3.106)

The phase is almost sinusoidal however with a pronounced asymmetry. As for the Fabry–Perot interferometer, we may distinguish between two scenarios: first, the round trip time being larger and, second, being smaller than the pulse duration. While the first case leads to a train of well-separated pulses, the second produces a single-phase-modulated pulse.

### 3.4.3 Pulse Shaping

All optical systems considered so far impose an amplitude and/or phase modulation on the pulse that is determined by either the dispersive properties of the material or the geometry of the optical system. In contrast, the so-called pulse shaping devices allow to generate almost arbitrary amplitude and/or phase modulations \([8]\). The most commonly used experimental setup is based on a zero dispersion compressor as shown in
Figure 3.7a. Its 4f arrangement may be thought of as a sequence of two 2f imaging setups and each of those performs a perfect optical Fourier transformation of the spatial wavepacket in the object plane. That is, the angular dispersion from the first grating is mapped to different locations at the so-called Fourier plane of the first lens. The second 2f arrangement annuls the first one and the original pulse is recovered. The actual phase modulator is a one-dimensional device and allows for an amplitude and/or phase modulation of each single-frequency component at the Fourier plane. Through computer programming the spatial light modulator, almost any linear transfer function can be realized.

3.5 Summary

In this chapter, we derived the one-dimensional propagation equation of time domain pulses through a dielectric material starting from Maxwell’s equations and a simple classical model describing the dielectric properties of the material. Going to a quantum description of the dielectric material often only requires to replace the classical oscillator strength by a quantum mechanical dipole transition element. The one-dimensional description of pulse propagation is accurate as long as the transverse beam profile is more or less constant from the point where the pulse enters the material to the point where it leaves the material. That is to say, either the Rayleigh length of the beam is larger than the thickness of the material or the transverse profile remains constant because the pulse is confined in a waveguide. We then discussed different aspects of the propagation equation, such as self-phase modulation, and so on by making educated simplifications. For very long propagation distances, for example, in optical fibers, we expanded the propagation equation by including the Raman process again through a simple classical model. The chapter was concluded with more complex optical systems, which are all designed to modify the temporal or spectral properties of a time domain pulse while leaving the transverse spatial beam profile unaffected.

References